Circuiti Logici Combinatori
Parte 1
Corso di Architettura degli Elaboratori (laboratorio)

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Overview

- Part 1 – Gate Circuits and Boolean Equations
  - Binary Logic and Gates
  - Boolean Algebra
  - Standard Forms
- Part 2 – Circuit Optimization
  - Two-Level Optimization
  - Map Manipulation
  - Practical Optimization (Espresso)
  - Multi-Level Circuit Optimization
- Part 3 – Additional Gates and Circuits
  - Other Gate Types
  - Exclusive-OR Operator and Gates
  - High-Impedance Outputs

Binary Logic and Gates

- **Binary variables** take on one of two values.
- **Logical operators** operate on binary values and binary variables.
- **Basic logical operators are the logic functions AND, OR and NOT.**
- **Logic gates** implement logic functions.
- **Boolean Algebra**: a useful mathematical system for specifying and transforming logic functions.
- **We study Boolean algebra as a foundation for designing and analyzing digital systems!**

Binary Variables

- Recall that the two binary values have different names:
  - True/False
  - On/Off
  - Yes/No
  - 1/0
- We use 1 and 0 to denote the two values.
- Variable identifier examples:
  - A, B, y, z, or X_1 for now
  - RESET, START_IT, or ADD1 later

Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT
- AND is denoted by a dot (·). (alt: $\wedge$)
- OR is denoted by a plus (+). (alt: $\vee$)
- NOT is denoted by an overbar (¯), a single quote mark (') after, or (~) before the variable.

Notation Examples

- **Examples:**
  - $Y = A \cdot B$ is read "Y is equal to A AND B."
  - $z = x + y$ is read "z is equal to x OR y."
  - $X = \overline{A}$ is read "X is equal to NOT A."
- **Note: The statement:**

  1 + 1 = 2 (read "one <u>plus</u> one equals two")

  is not the same as

  1 + 1 = 1 (read "1 <u>or</u> 1 equals 1").

Operator Definitions

- **Operations are defined on the values "0" and "1" for each operator:**

AND	OR	NOT
$0 \cdot 0 = 0$	$0 + 0 = 0$	$\overline{0} = 1$
$0 \cdot 1 = 0$	$0 + 1 = 1$	$\overline{1} = 0$
$1 \cdot 0 = 0$	$1 + 0 = 1$	
$1 \cdot 1 = 1$	$1 + 1 = 1$	

Truth Tables

- *Truth table* − a tabular listing of the values of a function for all possible combinations of values on its arguments
- Example: Truth tables for the basic logic operations:

AND		
X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

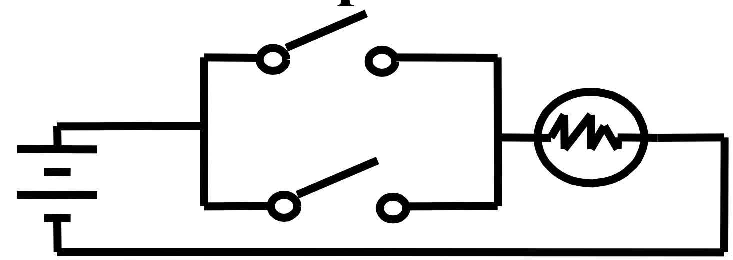
OR		
X	Y	$Z = X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT	
X	$Z = \overline{X}$
0	1
1	0

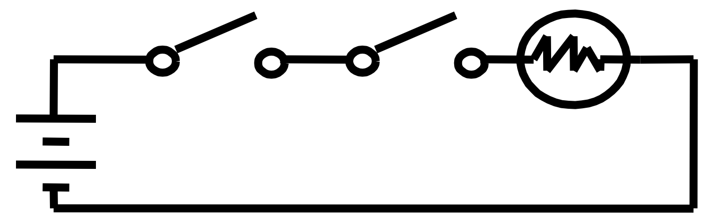
Logic Function Implementation

- **Using Switches**
  - **For inputs:**
    - **logic 1 is switch closed**
    - **logic 0 is switch open**
  - **For outputs:**
    - **logic 1 is light on**
    - **logic 0 is light off.**
  - **NOT uses a switch such that:**
    - **logic 1 is switch open**
    - **logic 0 is switch closed**

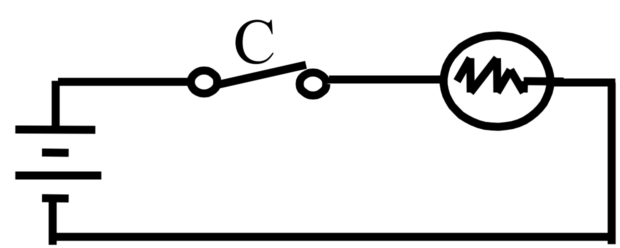
Switches in parallel => OR



Switches in series => AND

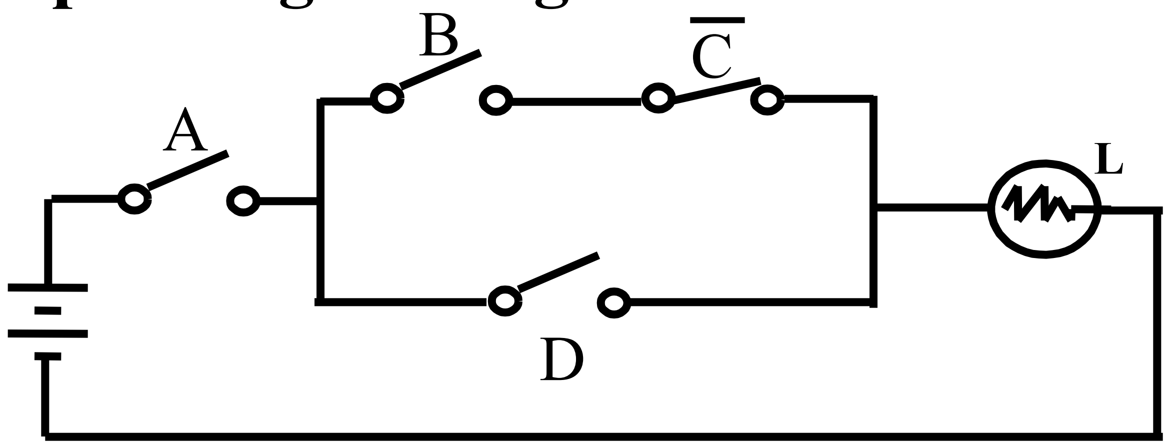


Normally-closed switch => NOT



Logic Function Implementation (Continued)

- **Example: Logic Using Switches**



- **Light is on (L = 1) for**

  L(A, B, C, D) = A ((B C') + D) = A B C' + A D

  and off (L = 0), otherwise.

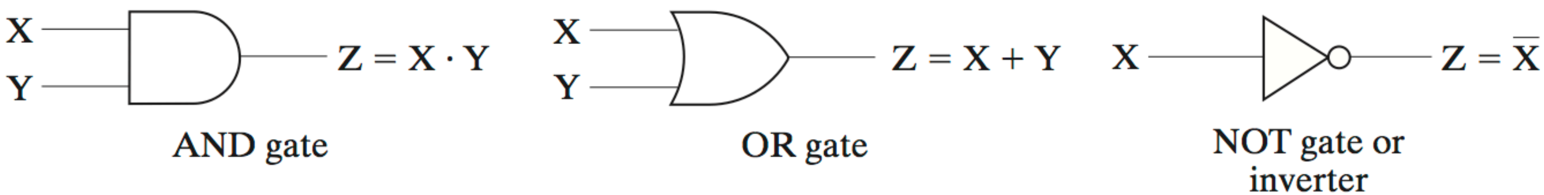
- **Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology**

Logic Gates

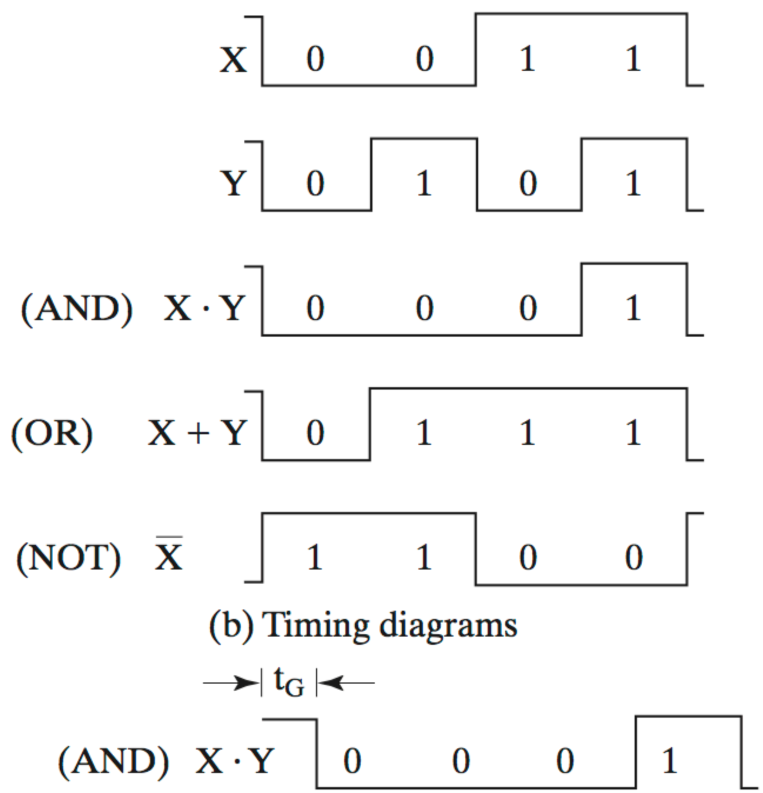
- **In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in *relays*. The switches in turn opened and closed the current paths.**
- **Later, *vacuum tubes* that open and close current paths electronically replaced relays.**
- **Today, *transistors* are used as electronic switches that open and close current paths.**

Logic Gate Symbols and Behavior

- **Logic gates have special symbols:**



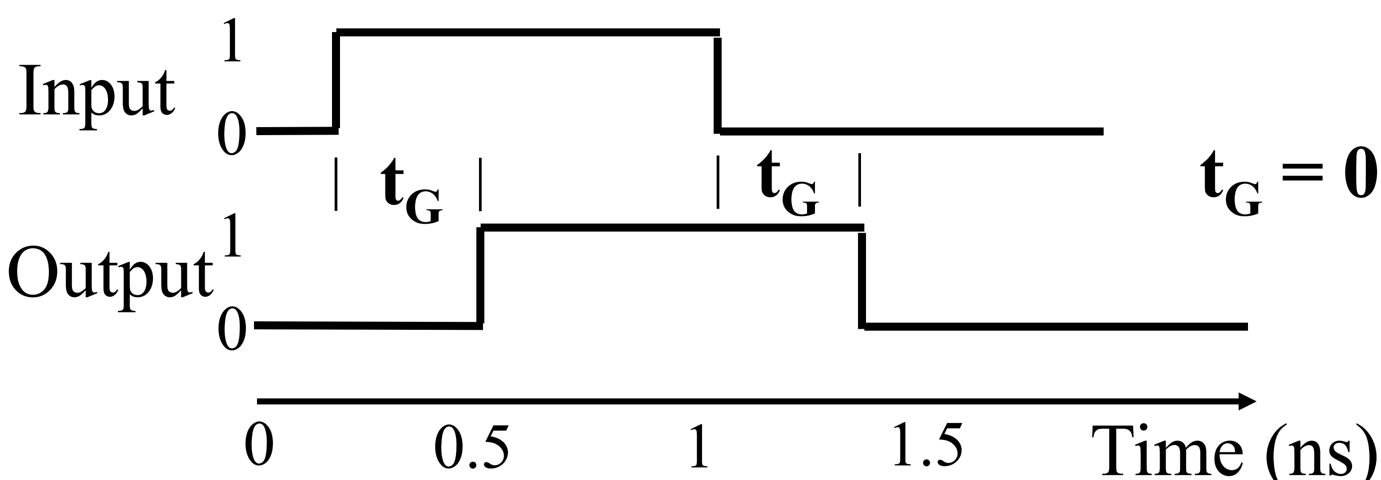
- **And waveform behavior in time as follows:**



(b) Timing diagrams

Gate Delay

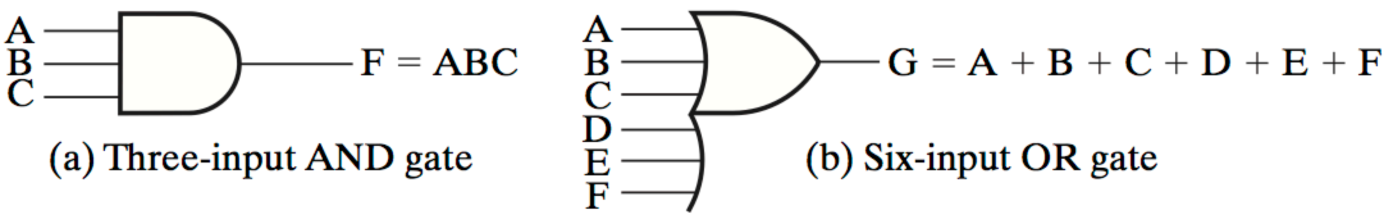
- **In actual physical gates, if one or more input changes causes the output to change, the output change does not occur instantaneously.**
- **The delay between an input change(s) and the resulting output change is the *gate delay* denoted by t_G:**



t_G = 0.3 ns

Logic Gate Symbols and Behavior

- *multi*-input gates are possible



(a) Three-input AND gate (b) Six-input OR gate

Logic Diagrams and Expressions

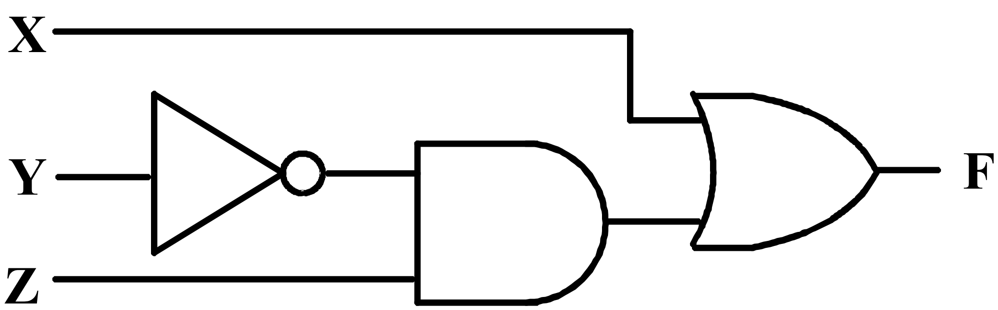
Truth Table

X Y Z	$F = X + \overline{Y} \cdot Z$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Equation

$$F = X + \overline{Y}\,Z$$

Logic Diagram



- Boolean equations, truth tables and logic diagrams describe the same function!
- Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.

Boolean Algebra

- **An algebraic structure defined on a set of at least two elements, B, together with three binary operators (denoted +, · and ‾) that satisfies the following basic identities:**

1.	$X + 0 = X$	2.	$X \cdot 1 = X$	
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	
5.	$X + X = X$	6.	$X \cdot X = X$	
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	
9.	$\overline{\overline{X}} = X$			
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$X + (Y + Z) = (X + Y) + Z$	13.	$X(YZ) = (XY)Z$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Some Properties of Identities & the Algebra

- If the meaning is unambiguous, we leave out the symbol "·"
- The identities above are organized into pairs. These pairs have names as follows:

1-4 Existence of 0 and 1	5-6 Idempotence
7-8 Existence of complement	9 Involution
10-11 Commutative Laws	12-13 Associative Laws
14-15 Distributive Laws	16-17 DeMorgan's Laws

- The <u>dual</u> of an algebraic expression is obtained by interchanging + and · and interchanging 0's and 1's.
- The identities appear in <u>dual</u> pairs. When there is only one identity on a line the identity is <u>self-dual</u>, i. e., the dual expression = the original expression.

Some Properties of Identities & the Algebra (Continued)

Truth tables to verify De Morgan's Theorem

A)	X	Y	$X+Y$	$\overline{X+Y}$	B)	X	Y	$\overline{X}$	$\overline{Y}$	$\overline{X} \cdot \overline{Y}$
	0	0	0	1		0	0	1	1	1
	0	1	1	0		0	1	1	0	0
	1	0	1	0		1	0	0	1	0
	1	1	1	0		1	1	0	0	0

Some Properties of Identities & the Algebra (Continued)

- **Unless it happens to be self-dual, the dual of an expression does not equal the expression itself.**
- **Example: $F = (A + \overline{C}) \cdot B + 0$**

  dual $F = (A \cdot \overline{C} + B) \cdot 1 = A \cdot \overline{C} + B$
- **Example: $G = X \cdot Y + (\overline{W + Z})$**

  dual $G =$ $((X+Y) \cdot (W \cdot Z)') = ((X+Y) \cdot (W' + Z')$
- **Example: $H = A \cdot B + A \cdot C + B \cdot C$**

  dual $H =$ $(A + B)(A + C)(B + C)$

  $= (A + BC)\ (B+C) = AB + AC + BC$
- **Are any of these functions self-dual?**

Some Properties of Identities & the Algebra (Continued)

- **There can be more that 2 elements in B, i. e., elements other than 1 and 0. What are some common useful Boolean algebras with more than 2 elements?**
  1. **Algebra of Sets**
  2. **Algebra of n-bit binary vectors**
- **If B contains only 1 and 0, then B is called the <u>switching algebra</u> which is the algebra we use most often.**

Boolean Operator Precedence

- **The order of evaluation in a Boolean expression is:**
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- **Consequence: Parentheses appear around OR expressions**
- **Example:** $F = A(B + C)(C + \overline{D})$

Example 1: Boolean Algebraic Proof

- $A + A \cdot B = A$ (Absorption Theorem)

Proof Steps	Justification (identity or theorem)
$A + A \cdot B$	
$= A \cdot 1 + A \cdot B$	$X = X \cdot 1$
$= A \cdot (1 + B)$	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$ (Distributive Law)
$= A \cdot 1$	$1 + X = 1$
$= A$	$X \cdot 1 = X$

- **Our primary reason for doing proofs is to learn:**
  - **Careful and efficient use of the identities and theorems of Boolean algebra, and**
  - **How to choose the appropriate identity or theorem to apply to make forward progress, irrespective of the application.**

Example 2: Boolean Algebraic Proofs

- $AB + \overline{A}C + BC = AB + \overline{A}C$ (Consensus Theorem)

Proof Steps

$AB + A'C + BC(A+A')$

$= AB + A'C + ABC + A'BC$

$= AB(1+C) + A'C(1+B)$

$= AB + A'C$

$(A+B)(A'+C)(B+C) = (A+B)(A'+C)$ (dual version)

Useful Theorems

- $x \cdot y + \overline{x} \cdot y = y \quad (x + y)(\overline{x} + y) = y$ **Minimization**
- $x + x \cdot y = x \quad x \cdot (x + y) = x$ **Absorption**
- $x + \overline{x} \cdot y = x + y \quad x \cdot (\overline{x} + y) = x \cdot y$ **Simplification**
- $x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$ **Consensus**

  $(x + y) \cdot (\overline{x} + z) \cdot (y + z) = (x + y) \cdot (\overline{x} + z)$
- $\overline{x + y} = \overline{x} \cdot \overline{y} \quad \overline{x \cdot y} = \overline{x} + \overline{y}$ **DeMorgan's Laws**

Proof of DeMorgan's Laws

$$\overline{x + y} = \overline{x} \cdot \overline{y} \qquad \overline{x \cdot y} = \overline{x} + \overline{y}$$

Boolean Function Evaluation

$F1 = xy\overline{z}$
$F2 = x + \overline{y}z$
$F3 = \overline{x}\,\overline{y}\,\overline{z} + \overline{x}yz + x\overline{y}$
$F4 = x\overline{y} + \overline{x}z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0		
0	0	1	0	1		
0	1	0	0	0		
0	1	1	0	0		
1	0	0	0	1		
1	0	1	0	1		
1	1	0	1	1		
1	1	1	0	1		

Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of <u>literals</u> (complemented and uncomplemented variables):

$$A\,B + \overline{A}\,C\,D + \overline{A}\,B\,D + \overline{A}\,C\,\overline{D} + A\,B\,C\,D$$

$$= AB + ABCD + \overline{A}\,C\,D + \overline{A}\,C\,\overline{D} + \overline{A}\,B\,D$$

$$= AB + AB(CD) + \overline{A}\,C\,(D + \overline{D}) + \overline{A}\,B\,D$$

$$= AB + \overline{A}\,C + \overline{A}\,B\,D = B(A + \overline{A}D) + \overline{A}C$$

$$= B\,(A + D) + \overline{A}\,C \quad \text{5 literals}$$

Expression Simplification

$F = X'YZ + X'YZ' + XZ$

$= X'Y(Z + Z') + XZ$

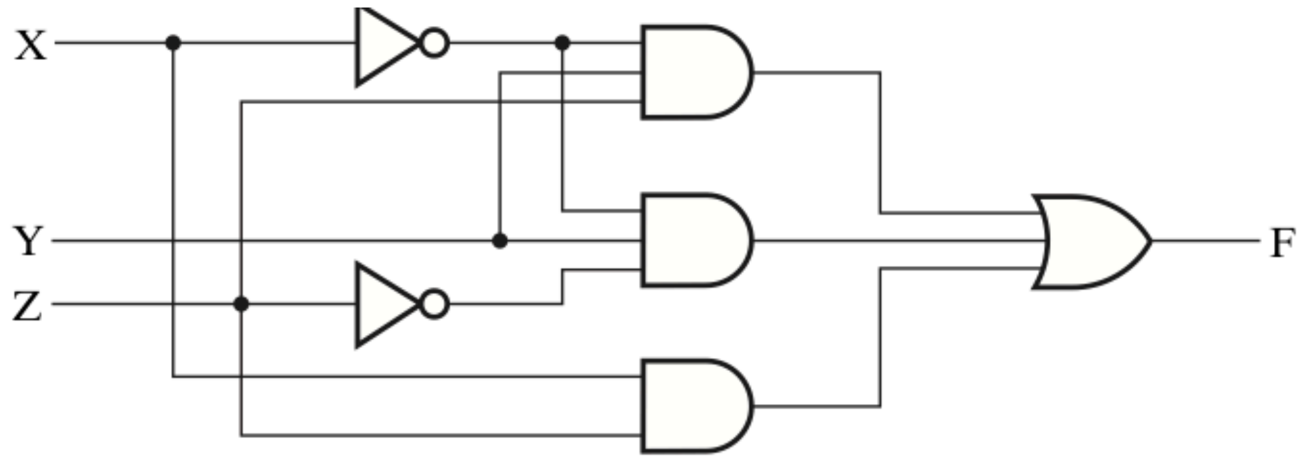
$= X'Y * 1 + XZ$

$= X'Y + XZ$

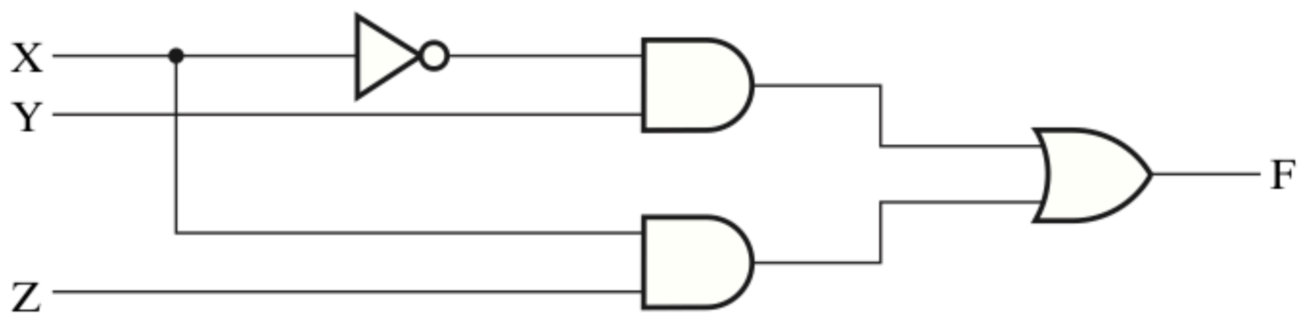
Truth Table for Boolean Function

X	Y	Z	(a) F	(b) F
0	0	0	0	0
0	0	1	0	0
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

Expression Simplification



(a) $F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$



(b) $F = \overline{X}Y + XZ$

Complementing Functions

- Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal
- Examples page 47!

Overview – Canonical Forms

- What are Canonical Forms?
- Minterms and Maxterms
- Index Representation of Minterms and Maxterms
- Sum-of-Minterm (SOM) Representations
- Product-of-Maxterm (POM) Representations
- Representation of Complements of Functions
- Conversions between Representations

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
  - Has a correspondence to the truth tables
- Canonical Forms in common usage:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)

Minterms

- <u>Minterms</u> are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for n variables.
- <u>Example:</u> Two variables (X and Y)produce 2 x 2 = 4 combinations:

  XY (both normal)
  $X\overline{Y}$ (X normal, Y complemented)
  $\overline{X}Y$ (X complemented, Y normal)
  $\overline{X}\,\overline{Y}$ (both complemented)

- Thus there are <u>four minterms</u> of two variables.

Maxterms

- **Maxterms are OR terms with every variable in true or complemented form.**
- **Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for *n* variables.**
- **Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:**

  $X + Y$ **(both normal)**

  $X + \overline{Y}$ **(x normal, y complemented)**

  $\overline{X} + Y$ **(x complemented, y normal)**

  $\overline{X} + \overline{Y}$ **(both complemented)**

Maxterms and Minterms

- Examples: Two variable minterms and maxterms.

Index	Minterm	Maxterm
0	$\overline{x}\,\overline{y}$	$x + y$
1	$\overline{x}\,y$	$x + \overline{y}$
2	$x\,\overline{y}$	$\overline{x} + y$
3	$x\,y$	$\overline{x} + \overline{y}$

- The index above is important for describing which variables in the terms are true and which are complemented.

Standard Order

- **Minterms and maxterms are designated with a subscript**
- **The subscript is a number, corresponding to a binary pattern**
- **The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.**
- **All variables will be present in a minterm or maxterm and will be listed in the same order (usually alphabetically)**
- **Example: For variables a, b, c:**
  - **Maxterms:** $(a + b + \bar{c})$, $(a + b + c)$
  - **Terms:** $(b + a + c)$, $a\,\bar{c}\,b$, and $(c + b + a)$ **are NOT in standard order.**
  - **Minterms:** $a\,\bar{b}\,c$, $a\,b\,c$, $\bar{a}\,\bar{b}\,c$
  - **Terms:** $(a + c)$, $\bar{b}\,c$, and $(\bar{a} + b)$ **do not contain all variables**

Purpose of the Index

- The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- For Minterms:
  - "1" means the variable is "Not Complemented" and
  - "0" means the variable is "Complemented".
- For Maxterms:
  - "0" means the variable is "Not Complemented" and
  - "1" means the variable is "Complemented".

Index Example in Three Variables

- Example: (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The <u>Index</u> 0 (base 10) = 000 (base 2) for three variables). All three variables are complemented for <u>minterm 0</u> ($\overline{X}, \overline{Y}, \overline{Z}$) and no variables are complemented for <u>Maxterm 0</u> (X,Y,Z).
  - Minterm 0, called m_0 is $\overline{X}\,\overline{Y}\,\overline{Z}$.
  - Maxterm 0, called M_0 is (X + Y + Z).
  - Minterm 6 ?
  - Maxterm 6 ?

Index Examples – Four Variables

Index i	Binary Pattern	Minterm m_i	Maxterm M_i
0	0000	$\bar{a}\,\bar{b}\,\bar{c}\,\bar{d}$	$a + b + c + d$
1	0001	$\bar{a}\,\bar{b}\,\bar{c}\,d$	?
3	0011	?	$a + b + \bar{c} + \bar{d}$
5	0101	$\bar{a}\,b\,\bar{c}\,d$	$a + \bar{b} + c + \bar{d}$
7	0111	?	$a + \bar{b} + \bar{c} + \bar{d}$
10	1010	$a\,\bar{b}\,c\,\bar{d}$	$\bar{a} + b + \bar{c} + d$
13	1101	$a\,b\,\bar{c}\,d$	?
15	1111	$a\,b\,c\,d$	$\bar{a} + \bar{b} + \bar{c} + \bar{d}$

Minterm and Maxterm Relationship

- Review: DeMorgan's Theorem
  $\overline{x \cdot y} = \overline{x} + \overline{y}$ and $\overline{x + y} = \overline{x} \cdot \overline{y}$
- Two-variable example:
  $M_2 = \overline{x} + y$ and $m_2 = x \cdot \overline{y}$
  Thus M_2 is the complement of m_2 and vice-versa.
- Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables
- giving:

$$M_i = \overline{m}_i \text{ and } m_i = \overline{M}_i$$

Thus M_i is the complement of m_i.

Function Tables for Both

- **Minterms of 2 variables**

x y	m_0	m_1	m_2	m_3
0 0	1	0	0	0
0 1	0	1	0	0
1 0	0	0	1	0
1 1	0	0	0	1

Maxterms of 2 variables

x y	M_0	M_1	M_2	M_3
0 0	0	1	1	1
0 1	1	0	1	1
1 0	1	1	0	1
1 1	1	1	1	0

- **Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i.**

Function Tables for Both

Minterms for Three Variables

X	Y	Z	Product Term	Symbol	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
0	0	0	$\overline{X}\,\overline{Y}\,\overline{Z}$	m_0	1	0	0	0	0	0	0	0
0	0	1	$\overline{X}\,\overline{Y}Z$	m_1	0	1	0	0	0	0	0	0
0	1	0	$\overline{X}Y\overline{Z}$	m_2	0	0	1	0	0	0	0	0
0	1	1	$\overline{X}YZ$	m_3	0	0	0	1	0	0	0	0
1	0	0	$X\overline{Y}\,\overline{Z}$	m_4	0	0	0	0	1	0	0	0
1	0	1	$X\overline{Y}Z$	m_5	0	0	0	0	0	1	0	0
1	1	0	$XY\overline{Z}$	m_6	0	0	0	0	0	0	1	0
1	1	1	XYZ	m_7	0	0	0	0	0	0	0	1

Maxterms for Three Variables

X	Y	Z	Sum Term	Symbol	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	$X+Y+Z$	M_0	0	1	1	1	1	1	1	1
0	0	1	$X+Y+\overline{Z}$	M_1	1	0	1	1	1	1	1	1
0	1	0	$X+\overline{Y}+Z$	M_2	1	1	0	1	1	1	1	1
0	1	1	$X+\overline{Y}+\overline{Z}$	M_3	1	1	1	0	1	1	1	1
1	0	0	$\overline{X}+Y+Z$	M_4	1	1	1	1	0	1	1	1
1	0	1	$\overline{X}+Y+\overline{Z}$	M_5	1	1	1	1	1	0	1	1
1	1	0	$\overline{X}+\overline{Y}+Z$	M_6	1	1	1	1	1	1	0	1
1	1	1	$\overline{X}+\overline{Y}+\overline{Z}$	M_7	1	1	1	1	1	1	1	0

Observations

- **In the function tables:**
  - **Each <u>min</u>term has one and only one 1 present in the 2^n terms (a <u>minimum</u> of 1s). All other entries are 0.**
  - **Each <u>max</u>term has one and only one 0 present in the 2^n terms All other entries are 1 (a <u>max</u>imum of 1s).**
- **We can implement any function by "OR-ing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.**
- **We can implement any function by "AND-ing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.**
- **This gives us two <u>canonical forms</u>:**
  - **<u>Sum of Minterms (SOM)</u>**
  - **<u>Product of Maxterms (POM)</u>**

for stating any Boolean function.

Minterm Function Example

- Example: Find $F_1 = m_1 + m_4 + m_7$
- $F1 = \bar{x}\ \bar{y}\ z + x\ \bar{y}\ \bar{z} + x\ y\ z$

x y z	index	m_1	+	m_4	+	m_7	$= F_1$
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$

Maxterm Function Example

- **Example: Implement F1 in maxterms:**

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z) \cdot (x + \bar{y} + z) \cdot (x + \bar{y} + \bar{z}) \cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F1$
0 0 0	0	0 · 1 · 1 · 1 · 1 = 0
0 0 1	1	1 · 1 · 1 · 1 · 1 = 1
0 1 0	2	1 · 0 · 1 · 1 · 1 = 0
0 1 1	3	1 · 1 · 0 · 1 · 1 = 0
1 0 0	4	1 · 1 · 1 · 1 · 1 = 1
1 0 1	5	1 · 1 · 1 · 0 · 1 = 0
1 1 0	6	1 · 1 · 1 · 1 · 0 = 0
1 1 1	7	1 · 1 · 1 · 1 · 1 = 1

Maxterm Function Example

- $F(A,B,C,D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- F(A, B,C,D) =

Canonical Sum of Minterms

- **Any Boolean function can be expressed as a <u>Sum of Minterms</u>.**
  - **For the function table, the <u>minterms</u> used are the terms corresponding to the 1's**
  - **For expressions, <u>expand</u> all terms first to explicitly list all minterms. Do this by "AND-ing" any term missing a variable v with a term ($v + \bar{v}$).**
- **Example: Implement $f = x + \bar{x}\,\bar{y}$ as a sum of minterms.**

First expand terms: $f = x(y + \bar{y}) + \bar{x}\,\bar{y}$

Then distribute terms: $f = xy + x\bar{y} + \bar{x}\,\bar{y}$

Express as sum of minterms: $f = m_3 + m_2 + m_0$

Another SOM Example

- **Example: $F = A + \overline{B}\,C$**
- **There are three variables, A, B, and C which we take to be the standard order.**
- **Expanding the terms with missing variables:**

$F = A(B + B')(C + C') + (A + A')\,B'\,C$

$= ABC + ABC' + AB'C + AB'C' + AB'C + A'B'C$

- **Collect terms (removing all but one of duplicate terms):** $ABC + ABC' + AB'C + AB'C' + A'B'C$
- **Express as SOM:** $m1 + m4 + m5 + m6 + m7$

Shorthand SOM Form

- **From the previous example, we started with:**

$$F = A + \overline{B}\,C$$

- **We ended up with:**

$$F = m_1 + m_4 + m_5 + m_6 + m_7$$

- **This can be denoted in the formal shorthand:**

$$F(A, B, C) = \Sigma_m(1,4,5,6,7)$$

- **Note that we explicitly show the standard variables in order and drop the "m" designators.**

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a Product of Maxterms (POM).
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law , "OR-ing" terms missing variable v with a term equal to $\mathbf{v} \cdot \overline{\mathbf{v}}$ and then applying the distributive law again.
- Example: Convert to product of maxterms:

$$f(x, y, z) = x + \overline{x}\,\overline{y}$$

Apply the distributive law:

$$x + \overline{x}\,\overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y}$$

Add missing variable z:

$$x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- Convert to Product of Maxterms:

$$f(A,B,C) = A\overline{C} + BC + \overline{A}\,\overline{B}$$

- Use $x + yz = (x+y)\cdot(x+z)$ with $x = (A\overline{C} + BC)$, $y = \overline{A}$, and $z = \overline{B}$ to get:

$$f = (A\overline{C} + BC + \overline{A})(A\overline{C} + BC + \overline{B})$$

- Then use $x + \overline{x}y = x + y$ to get:

$$f = (\overline{C} + BC + \overline{A})(A\overline{C} + C + \overline{B})$$

and a second time to get:

$$f = (\overline{C} + B + \overline{A})(A + C + \overline{B})$$

- Rearrange to standard order,

$$f = (\overline{A} + B + \overline{C})(A + \overline{B} + C) \text{ to give } f = M_5 \cdot M_2$$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x, y, z) = \Sigma_m(1,3,5,7)$

$$\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$$

$$\overline{F}(x, y, z) = \Pi_M(1,3,5,7)$$

Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.
- Example:Given F as before: $F(x, y, z) = \Sigma_m(1,3,5,7)$
- Form the Complement: $\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $F(x, y, z) = \Pi_M(0,2,4,6)$

Standard Forms

- **Standard Sum-of-Products (SOP) form:** equations are written as an OR of AND terms
- **Standard Product-of-Sums (POS) form:** equations are written as an AND of OR terms
- **Examples:**
  - SOP: $A\,B\,C + \overline{A}\,\overline{B}\,C + B$
  - POS: $(A + B) \cdot (A + \overline{B} + \overline{C}) \cdot C$
- These "mixed" forms are **neither SOP nor POS**
  - $(A\,B + C)\,(A + C)$
  - $A\,B\,\overline{C} + A\,C\,(A + B)$

Standard Sum-of-Products (SOP)

- **A sum of minterms form for *n* variables can be written down directly from a truth table.**
  - **Implementation of this form is a two-level network of gates such that:**
  - **The first level consists of *n*-input AND gates, and**
  - **The second level is a single OR gate (with fewer than 2^n inputs).**
- **This form often can be simplified so that the corresponding circuit is simpler.**

Standard Sum-of-Products (SOP)

- **A Simplification Example:**
- $F(A,B,C) = \Sigma m(1,4,5,6,7)$
- **Writing the minterm expression:**

$$F = \bar{A}\,\bar{B}\,C + A\,\bar{B}\,\bar{C} + A\,\bar{B}\,C + AB\bar{C} + ABC$$

- **Simplifying:**

$F = A'B'C + A\,(B'C' + BC' + B'C + BC)$

$= A'B'C + A\,(B' + B)\,(C' + C)$

$= A'B'C + A \cdot 1 \cdot 1$

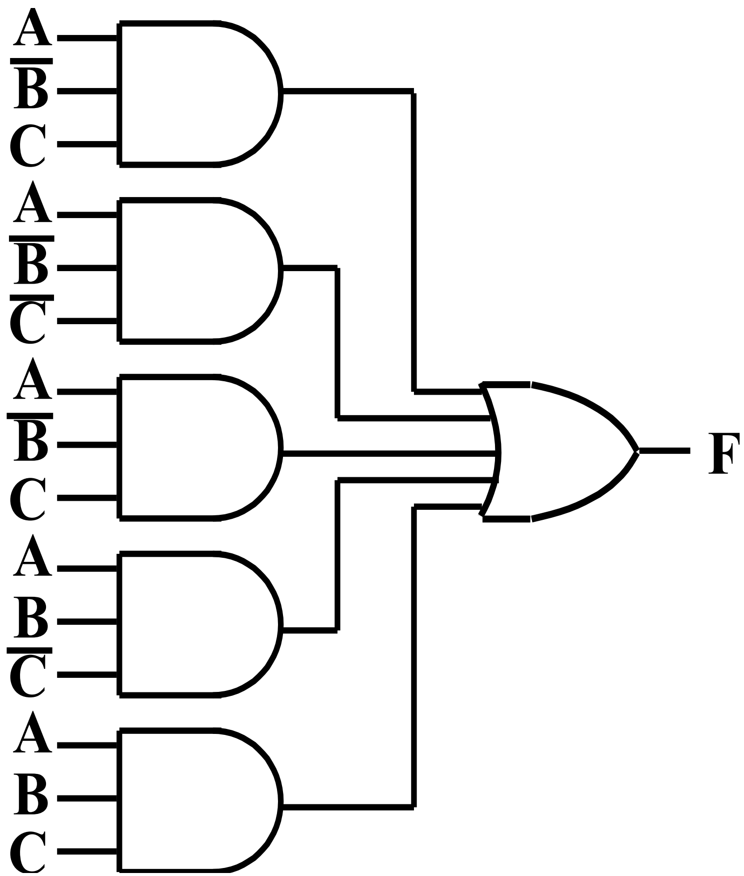
$= A'B'C + A$

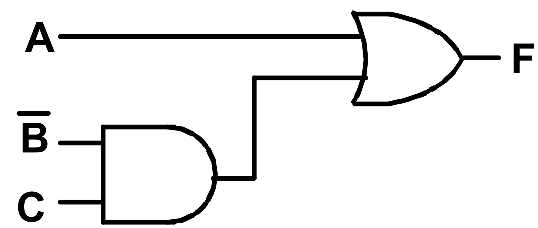
$= B'C + A$

- **Simplified F contains 3 literals compared to 15 in minterm F**

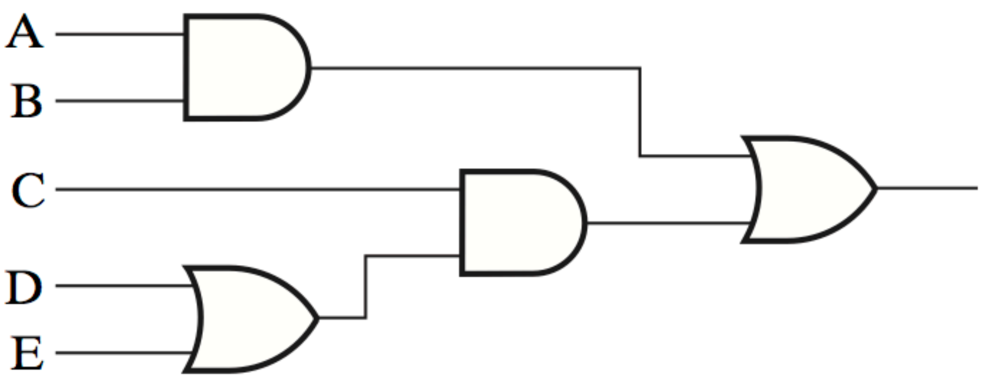
AND/OR Two-level Implementation of SOP Expression

- **The two implementations for F are shown below – it is quite apparent which is simpler!**

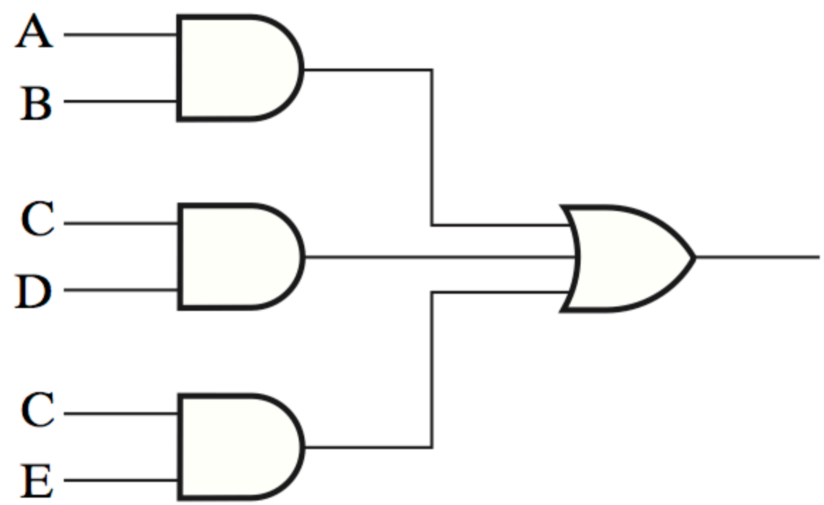




AND/OR Two-level Implementation of SOP Expression



(a) AB + C(D + E)



(b) AB + CD + CE

SOP and POS Observations

- **The previous examples show that:**
  - **Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity**
  - **Boolean algebra can be used to manipulate equations into simpler forms.**
  - **Simpler equations lead to simpler two-level implementations**
- **Questions:**
  - **How can we attain a "simplest" expression?**
  - **Is there only one minimum cost circuit?**
  - **The next part will deal with these issues.**

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